Exercise 2.1 Solution

The Proof follows the ones of Prop 2.2 and Cor 2.4:

$$\mathbb{E}[\hat{R}_\varphi(\hat\theta)] = \mathbb{E}\left[\frac{1}{m}\|y-\varphi\hat\theta\|^2\right] = \mathbb{E}\left[\frac{1}{m}\|y-\varphi\theta^* + \varphi(\theta^*-\theta)\|^2\right]$$

$$= \mathbb{E}\left[\frac{1}{m}\|y-\varphi\theta^*\|^2 + \frac{2}{m}(y-\varphi\theta^*)^T\varphi(\theta^*-\theta) + \frac{1}{m}\|\varphi(\theta^*-\theta)\|_2^2\right]$$

$$= \underbrace{R_\varphi(\theta)}_{\sigma^2} + \underbrace{\mathbb{E}\left[\|\hat\theta-\theta^*\|_\Sigma^2\right]}_{\frac{\sigma^2 d}{m} \text{ by Cor 2.4}} + \frac{2}{m}\underbrace{\mathbb{E}\left[(y-\varphi\theta^*)^T\varphi(\theta^*-\hat\theta)\right]}$$

This was equals to zero in the class because $\hat\theta \perp\!\!\!\perp y$ but this is not the case for the empirical risk because $\hat\theta$ and y use the same data.

Using $y = \varphi\theta^* + \varepsilon$, we have

$$\mathbb{E}\left[(y-\varphi\theta^*)^T\varphi(\theta^*-\hat\theta)\right] = \underbrace{\mathbb{E}[\varepsilon^T\varphi\theta^*]}_{=0} - \mathbb{E}[\varepsilon^T\varphi\hat\theta]$$

since $\hat\theta = (\varphi^T\varphi)^{-1}\varphi^T y$

$$= -\mathbb{E}\left[\varepsilon^T\varphi(\varphi^T\varphi)^{-1}\varphi^T y\right]$$

$$= -\mathbb{E}\left[\varepsilon^T\varphi(\varphi^T\varphi)^{-1}\varphi^T\varepsilon\right]$$

$$= -\mathbb{E}\left[\mathrm{Tr}\left(\varepsilon^T\varphi(\varphi^T\varphi)^{-1}\varphi^T\varepsilon\right)\right]$$

$$= -\mathbb{E}\left[\mathrm{Tr}\left(\varphi(\varphi^T\varphi)^{-1}\varphi^T\varepsilon\varepsilon^T\right)\right]$$

$$= -\mathrm{Tr}\left(\varphi(\varphi^T\varphi)^{-1}\varphi^T\,\mathbb{E}[\varepsilon\varepsilon^T]\right)$$

$$= -\mathrm{Tr}\left(\varphi(\varphi^T\varphi)^{-1}\varphi^T(\sigma^2 I_m)\right)$$

$$= -\sigma^2\,\mathrm{Tr}\left(\varphi(\varphi^T\varphi)^{-1}\varphi^T\right)$$

$$= -\sigma^2 d$$

Thus $\mathbb{E}[\hat{R}_\varphi(\hat\theta)] = \sigma^2 + \frac{\sigma^2 d}{m} - \frac{2\sigma^2 d}{m} = \frac{m-d}{m}\sigma^2$